[redacted]

N67-86856
(ACCESSION NUMBER)

42
(PAGES)

CR#88985
(NASA CR OR TMX OR AD NUMBER)

(THRU)

(CODE)

99
(CATEGORY)

FACILITY FORM 602

ICSC

INTERNATIONAL CONSULTANT SCIENTISTS Corporation

THEORETICAL AND EXPERIMENTAL STUDIES OF THE DEFORMATION OF THIN SHELLS

Final Report

Contract NASw-622

Submitted to:

National Aeronautics and Space Administration

10 January 1965

International Consultant Scientists Corporation
127 Winthrop Road
Brookline 46, Massachusetts

THEORETICAL AND EXPERIMENTAL STUDIES OF THE DEFORMATION OF THIN SHELLS.

ABSTRACT

In this study, we have looked for a method to ease the practical use of the theory of thin shells. Ultimately, a correspondance between curved shells and appropriately chosen planar shells is established.

Our first step was to determine which were the best equations on which to base our developments. We know that various methods, which ultimately lead to analogous equations, have been suggested. We have tried to compare these various methods by examing to what degree of approximations they culminate when we consider various terms that are often neglected. After numerous trials, we have arrived at a method of calculation which defines the meridian by means of the relation, $\Phi = f(s)$ between the arc s of the meridian and the angle Φ between the normal and the axis. By the use of examples, we show that this method allows us to study, through relatively simple calculations, profiles for which the usual methods break down.

We then approached the problem of the correspondance between a curved shell of constant thickness and a planar shell whose thickness varies as a function of the distance to the center of the shell. We tried to equate the principal constraints of the two shells which leads to a correspondance between their parallels and to a convenient law for the variation of the thickness of the planar shell. Next, choosing convenient boundary conditions, we verified that the moments were of the same order of magnitude in both cases. As a result, it becomes possible to ease the construction of an ogive shell, by experimenting on a planar disc whose variation in thickness will be chosen such as to diminish the maximum values of the constraints.

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DEVELOPMENTS ON THE THEORY OF THIN SHELLS.

The theory of thin shells is among those in the limelight of contemporary interest and which are still preoccupying the scientists of today. We need not recall in detail all the theories which have emerged on this subject. Furthermore, we all know how and why the hypotheses concerning the small thickness of the shell result both in simplifications and in the need of approximations whose validity, or lack of it, is not always as readily apparent as might be desired.

In this work, we shall deal only with shells of revolution about an axis, all conditions being those of revolution about this axis.

If we refer to the book of W. Flugge, Stresses in Shells, (Springer, 1960), we see how we can write equations defining the constraints on the median surface of the shell, by neglecting higher order terms. By considerations which we shall not reproduce here but which can be found starting on page 320 of the above referred work, we arrive at the following formulae:

$$\frac{\partial}{\partial\phi}(rN_\phi) - R_1 N_\theta \cos\phi - rQ_\phi = -rR_1 p_\phi$$

$$\frac{\partial}{\partial\phi}(rQ_\phi) + R_1 N_\theta \sin\phi + rN_\phi = rR_1 p_r$$

$$\frac{\partial}{\partial\phi}(rM_\phi) - R_1 M_\theta \cos\phi = rR_1 Q_\phi$$

The N's and the M's are related to the displacements by the equations:

$$N_\phi = D\left[\frac{\frac{\partial v}{\partial\phi} + w}{R_1} + \nu\,\frac{v\cos\phi + w\sin\phi}{r}\right]$$

$$N_\theta = D\left[\frac{v\cos\phi + w\sin\phi}{r} + \nu\,\frac{\frac{\partial v}{\partial\phi} + w}{R_1}\right]$$

$$M_\phi = \frac{K}{R_1}\frac{\partial}{\partial\phi}\left(\frac{\partial w/\partial\phi}{R_1}\right) + \nu \frac{\frac{\partial w}{\partial\phi}\cos\phi}{r}$$

$$M_\theta = \frac{K}{R_1}\frac{\frac{\partial w}{\partial\phi}\cos\phi}{r} + \nu\frac{\partial}{\partial\phi}\left(\frac{\partial w/\partial\phi}{R_1}\right)$$

In these formulas, the numbers D and K represent,

$$D = \frac{Eh}{1-\nu^2} \quad ; \quad K = \frac{Eh^3}{12(1-\nu^2)}$$

where h is the thickness, for the moment assumed to be constant, of the shell. r and θ are the polar coordinates in the plane of a parallel on the median surface and ϕ is the angle between the normal to this surface and the axis of revolution. The N's and the M's are components of the constraints and the moments with respect to the usual axes. The Q, s are the transversal forces; p will denote the external pressure. u, v, and w are the components of inifinitesimal deformations undergone by the shell in the direction of the parallel, the meridian, and the normal to the surface, respectively. R_1 and R_2 designate the principal radii of curvature.

Flugge shows (op. cit., p321 to 353) how to eliminate the terms in p without introducing appreciable error.

Further, (pp357 and following) by letting,

$$\chi = \frac{\frac{\partial w}{\partial\phi} - v}{R_1}$$

the equations concerning the M's become,

$$M_\phi = K\left[\frac{1}{R_1}\frac{\partial\chi}{\partial\phi} + \nu\frac{\chi}{R_2}\cot\phi\right]$$

$$M_\theta = K\left[\frac{\chi}{R_2}\cot\phi + \nu\frac{1}{R_1}\frac{\partial\chi}{\partial\phi}\right]$$

These formulas correspond to the equations established by Flugge on p. 320 (op. cit.) with the terms in $R_1 - R_2$ neglected, that is:

$$\text{for } M_\phi: \frac{K}{R_1^2 R_2}\left(\frac{\delta v}{\partial \phi} + w\right)(R_2 - R_1)$$

$$\text{for } M_\theta: -\frac{K}{R_1 R_2^2}(v\cot\phi + w)(R_2 - R_1)$$

Since we have,

$$\varepsilon_\phi = \frac{\frac{\partial v}{\partial \phi} + w}{R_1} \qquad ; \qquad \varepsilon_\theta = \frac{v\cot\phi + w}{R_2}$$

where the ε's are the usual components of strains, we see that in general the terms in question are small with respect to the terms retained.

The consequences shown in subsequent paragraphs (op. cit) validate this hypothesis. Thus we see that various suppostions have been made and that the ultimate consequences of the approximations are not evident. We shall return to this point in due time. However, for now, let us notice that in the case of a conical shell, the expressions neglected above in M are actually,

$$\text{for } M_\phi: \; 0$$

$$\text{for } M_\theta: \; K(v\cot\phi + w)\frac{\sin^2\phi}{r^2}$$

Before going any further, we shall show another way of approaching the problem of the deformation of thin shells. This method is based on the following considerations: (ref. Salet G., <u>Bulletin de la Societe Francaise des Mecaniciens</u>, June 1951, p. 17.) Suppose the meridian of the shell to be studied is defined by the equation, $\phi = f(s)$, which relates the arc s to the angle before any deformations occur. If a small deformation takes place, the above relation must be replaced by , $\phi = f(s) + g(s)$, such that the increase $\delta\phi$ is given by, $\delta\phi = g(s)$. We note that the curvature,

$$\frac{1}{R_1} = \frac{d\phi}{ds} = f'(s)$$

of the meridian will have experienced an increase

$$\delta\left(\frac{1}{R_1}\right) = g'(s)$$

With this defined, still considering the case of revolution, we arrive at the following formulas for the M's (Salet, op. cit., p.19):

$$M_\phi = K\left[\delta\left(\frac{1}{R_2}\right) + \nu\,\delta\left(\frac{1}{R_1}\right)\right]$$

$$M_\theta = K\left[\delta\left(\frac{1}{R_1}\right) + \nu\,\delta\left(\frac{1}{R_2}\right)\right]$$

The increase $\delta\left(\frac{1}{R_2}\right)$ which appears in these formulase is easily found to be equal to,

$$\delta\left(\frac{1}{R_2}\right) = \frac{\cos\phi}{r}\,g - \frac{\sin\phi}{r}\,\frac{\delta r}{r}$$

The author adopts the following approximation,

$$\delta\left(\frac{1}{R_2}\right) = \frac{\cos\phi}{r}\,g$$

neglecting the second term with respect to the first by reason of the argument that follows.

We note first that a unit of expansion $\frac{\delta r}{r}$ along a paralle-- of the surface is equal to

$$\frac{1}{E}\left(N_\theta - \nu N_\phi\right)$$

However, if S is the limit of the elasticity of the metal, it is obvious that $N_\theta - \nu N_\phi$ must not exceed S; thus we must have,

$$\left|\frac{\delta r}{r}\right| < \frac{S}{E}$$

The absolute value of the term under consideration is then less than $\frac{S}{E}\frac{\sin\phi}{r}$. Corresponding to it, we have in the moments M_ϕ and M_θ increases which themselves correspond to increases in the stresses on the surface, equal to, $\frac{Eh}{2}\delta\left(\frac{1}{R_2}\right) < \frac{Sh}{2R_2}$.

Since the thickness h is very small with respect to the radius of curvature R_2, we can conclude that the term in question can be neglected.

The preceding considerations deserve a verification. We must ultimately determine if the quantity

$$\frac{\sin\phi}{r}\,\frac{\delta r}{r} = \frac{1}{E}(N_\theta - \nu N_\phi)\frac{\sin\phi}{r} = \frac{N_\theta - \nu N_\phi}{ER_2}$$

can be neglected with respect to $\frac{\cos\phi}{r}\,g$.

Such as it may be, for the time being, the calculations (Salet op. cit., p. 20) show that we obtain the following equation for the bending moments M_ϕ and M_θ, for the shearing stress t', and for the constraints N_ϕ and N_θ :

$$M_\phi = \frac{Eh^3}{12(1-\nu^2)}\left[\frac{\cos\phi}{r}\,g + \nu\, g'\right]$$

$$M_\theta = \frac{Eh^3}{12(1-\nu^2)}\left[g' + \nu\frac{\cos\phi}{r}\,g\right]$$

$$t = \frac{Eh^2}{12(1-\nu^2)}\left[g'' + g'\frac{\cos\phi}{r} - \left(\frac{\cos^2\phi}{r^2} + \nu\frac{\sin\phi}{r}f'\right)g\right]$$

$$N_\phi = \frac{pr}{2h\sin\phi} + \frac{Eh^2}{12(1-\nu^2)}\left[g''\cot\phi + \frac{1}{r}\frac{\cos^2\phi}{\sin\phi}g' - \left(\frac{\cos^3\phi}{\sin\phi}\frac{1}{r^2} + \nu\cos\phi\frac{f'}{r}\right)g\right]$$

$$N_\theta = \frac{pr}{h\sin\phi}\left(1 - \frac{rf'}{2\sin\phi}\right) + \frac{Eh^2}{12(1-\nu^2)}\left\{\begin{aligned}&\frac{r}{\sin\phi}g''' + \left(2\cot\phi - \frac{\cos\phi}{\sin^2\phi}rf'\right)g''\\ &-\left[\left(\nu + \frac{1}{\sin^2\phi}\right)f' + \frac{1}{r}\frac{\cos^2\phi}{\sin\phi}\right]g'\\ &+\left[\left(\cos\phi + \frac{\cos\phi}{\sin^2\phi}\right)\frac{f'}{r} + \frac{\cos^3\phi}{r^2\sin\phi} - \nu f''\right]g\end{aligned}\right\}$$

The function g(s) which defines the deformation of the meridian satisfies the differential equation,

$$g^{IV} = ag''' + bg'' + cg' + dg + k$$

where the coefficients a, b, c, d, and k have the following values:

$$a = 2f'\cot\phi - 4\frac{\cos\phi}{r}$$

$$b = \frac{5 - 2\sin^2\phi}{\sin\phi}\frac{f'}{r} + f''\cot\phi - \frac{1+\cos^2\phi}{\sin^2\phi}f'^2$$

$$c = (-\cot\phi - 2\sin\phi\cos\phi)\frac{f'}{r^2} + \left(\frac{1}{\sin\phi} + 2\nu\sin\phi\right)\frac{f''}{r} - \frac{2\cos\phi}{\sin^2\phi}\frac{f'^2}{r}$$

$$d = \begin{aligned}&-12(1-\nu^2)\frac{\sin^2\phi}{h^2r^2} + \frac{\cos^2\phi}{\sin\phi}(1+2\sin^2\phi)\frac{f'}{r^3} + \left[(1+\nu^2)\sin^2\phi + \frac{1+\cos^2\phi}{\sin^2\phi}\right]\frac{f'^2}{r}\\ &+ \left[(\nu-1)\sin\phi\cos\phi - \cot\phi\right]\frac{f''}{r^2} + \nu\frac{\sin\phi}{r}f'''\end{aligned}$$

$$k = \frac{12p(1-\nu^2)}{Eh^3}\frac{r}{2\sin\phi}\left[f'' + \frac{5}{2}\cot\phi\, f' - r\frac{\cot\phi}{\sin\phi}f'^2 - \frac{3}{2}\frac{\cos\phi}{r}\right]$$

We shall now examine particular cases in order to more effectively verify the quality of the approximations which have been made in methods used. We shall consider the case of a spherical shell and that of a conical shell. These two cases are treated in an analogous manner in the theory of Flugge.

Let us recall that in the case where the shell is a portion of a sphere of radius R, the equations, of equilibrium take the form

$$\begin{aligned}\frac{\partial(N_\phi\sin\phi)}{\partial\phi} - N_\theta\cos\phi + Q_\phi\sin\phi &= 0\\ \frac{\partial(Q_\phi\sin\phi)}{\partial\phi} + N_\theta\sin\phi + N_\phi\sin\phi &= 0 \qquad (1)\\ \frac{\partial(M_\phi\sin\phi)}{\partial\phi} - M_\theta\cos\phi - RQ_\phi\sin\phi &= 0\end{aligned}$$

where the constraints M and N are defined as a function of the small deformations u, v, and w by the following equations:

$$N_\phi = \frac{Eh}{R(1-\nu^2)}\left[w + \frac{\partial v}{\partial \phi} + \nu\,(w + v\cot\phi)\right]$$

$$N_\theta = \frac{Eh}{R(1-\nu^2)}\left[w + v\cot\phi + \nu\,(w + \frac{\partial v}{\partial \phi})\right]$$

$$M_\phi = \frac{Eh^3}{12R^2(1-\nu^2)}\left[\frac{\partial}{\partial\phi}(\frac{\partial w}{\partial \phi} - v) + \nu\cot\phi\,(\frac{\partial w}{\partial\phi} - v)\right]$$

$$M_\theta = \frac{Eh^3}{12R^2(1-\nu^2)}\left[(\frac{\partial w}{\partial\phi} - v)\cot\phi + \nu\,\frac{\partial}{\partial\phi}(\frac{\partial w}{\partial\phi} - v)\right]$$

The above expressions should be substituted in the equilibrium equations given on the preceding page.

We immediately see that the combination,

$$\beta = \frac{1}{R}\,(\frac{\partial w}{\partial\phi} - v)$$

plays a dominant role here. The values of M can be expressed as functions of β , and substituting into the third equation of group (1) we have:

$$\frac{\partial^2\beta}{\partial\phi^2} + \frac{\partial\beta}{\partial\phi}\cot\phi - \beta\,(\cot^2\phi + \nu) = \frac{12R^2(1-\nu^2)}{Eh^3}\,Q_\phi$$

This constitutes the first relation between β and Q_ϕ. We shall obtain a second one by the following process.

The equations defining the N's are equivalent to,

$$w + \frac{\partial v}{\partial\phi} = \frac{R(1-\nu^2)}{Eh}(N_\phi - \nu\,N_\theta)$$

$$w + v\cot\phi = \frac{R(1-\nu^2)}{Eh}\,(N_\theta - \nu\,N_\phi)$$

Let us differentiate the second of these equations with respect to ϕ; we then have three equations linear in $\frac{\partial v}{\partial\phi}$, w, $\frac{\partial w}{\partial\phi}$ as functions of v, N_ϕ, and N_θ . Then,

$$R\beta = \frac{\partial w}{\partial \phi} - v = \frac{R(1-\nu^2)}{Eh}\left[\frac{\partial N_\theta}{\partial \phi} - \nu \frac{\partial N_\phi}{\partial \phi} + (1+\nu)(N_\theta - N_\phi)\cot\phi\right] \quad (2)$$

However, the second equation of (1) gives,

$$N_\theta = -N_\phi - \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(Q_\phi \sin\phi)$$

and the first equation of (1) consequently becomes,

$$\frac{\partial}{\partial\phi}(N_\phi \sin^2\phi) + \frac{\partial}{\partial\phi}(Q_\phi \sin\phi\cos\phi) = 0$$

We conclude

$$N_\phi \sin\phi + Q_\phi \cos\phi = \frac{K}{\sin\phi}$$

where K is a constant. From this we obtain N_ϕ, and consequently N_θ from the preceding relation. Substituting these values of N into equation (2), we have a second relation between β and Q_ϕ,

$$\frac{\partial^2 Q_\phi}{\partial\phi^2} + \cot\phi\,\frac{\partial Q_\phi}{\partial\phi} - (\cot^2\phi - \nu)Q_\phi = -\frac{Eh}{1-\nu^2}\beta \qquad (3)$$

The details of these calculations can be found in the work of W. Flugge. The essence is as follows.

Let us introduce the operator,

$$L(\ ) = \frac{\partial^2}{\partial\phi^2}(\ ) + \cot\phi\,\frac{\partial}{\partial\phi}(\ ) - (\ )\cot^2\phi$$

Our two equations in β and Q_ϕ become,

$$L(\beta) - \nu\beta = \frac{12R^2(1-\nu^2)}{Eh^3}Q_\phi$$

$$L(Q_\phi) + \nu Q_\phi = -\frac{Eh}{1-\nu^2}\beta$$

Letting

$$\lambda^4 = \frac{3R^2}{h^2} - \frac{\nu^2}{4}$$

we have,

$$LL(Q_\phi) + 4\lambda^4 Q_\phi = 0 \qquad (4)$$

Thus we are left with equation (4) to solve and then substituting into formula (3) we obtain β . As Flugge has noted, equation can also be written as

$$L[L(\Omega_\phi) + 2\varepsilon i\lambda^2 \Omega_\phi] - 2\varepsilon i\lambda^2 [L(\Omega_\phi) + 2\varepsilon i\lambda^2 \Omega_\phi] = 0$$

where $\varepsilon = \pm 1$. This last equation has two particular solutions which are themselves second order equations of the form,

$$L(\Omega) + 2\varepsilon i\lambda^2 \Omega = 0$$

It is enough to consider only one of these second order equations since the solution of this equation, which has imaginary coefficients, are themselves imaginary; consequently the imaginary part and the real part are separately solutions of equation . This latter being linear, we then have four particular solutions. It is sufficient to make a linear combination of these.

The problem has been reduced to solving the equation,

$$L(\Omega) + 2\varepsilon i\lambda^2 \Omega = 0$$

whcih, by a change of variables

$$x = \cos^2\phi \qquad \Omega = z\sin\phi$$

becomes a hypergeometric equation,

$$\frac{d^2z}{dx^2} + \frac{1-5x}{2x(1-x)}\frac{dz}{dx} - \frac{1-2i\lambda^2}{4x(1-x)} z = 0$$

We shall not reproduce here the details of these calculations which can be found on page 325 of the work of Flugge (op. cit.) Now, we can determine, by means of the explicit formulas thus defined, whether the approximation of which we spoke previously is valid.

If we adopt the method of calculation indicated on page 3 of this report, we shall very simply find the following values for the coefficients a, b, c, d, and k:

$$a = -\frac{2\cot\phi}{R}; \quad b = \frac{3-\sin^2\phi}{R^2\sin^2\phi}; \quad c = -\frac{\cos\phi}{R^3\sin^3\phi}(3+2\sin^2\phi)$$

$$d = -\frac{12(1-\nu)}{h^2R^2} + \frac{\cos^2\phi(3+\sin^2\phi) + \nu^2\sin^4\phi}{R^4\sin^4\phi}$$

$$k = -\frac{18p(1-\nu^2)\cot\phi}{Eh^3R}$$

The corresponding equation in g is then a fourth order linear differential equation which is easy to solve numerically by known methods. Then, without difficulty, we can proceed to verify the approximation stated on page 5 . This verification simply demands fairly long numerical computations whcih can be performed on computers. The result will be indicated later.

Let us now examine a second particular case, that of a cone of revolution, again requiring symmetry of revolution around its axis. Under these conditions, the angle ϕ between the normal to the surface and the axis remains constant,, equal to α , and we must replace this variable by the arc s of the meridian as measured on a ray of the cone. The elementary formulae,

$$ds = R_1 d\phi \ , \quad r = s\cos\alpha, \quad R_2 = s\cot\alpha \ ,$$

of which the first holds even before supposing ϕ to be constant, immediately give the following formulae for the equilibrium conditions:

$$\begin{aligned} \frac{d(sN_\phi)}{ds} - N_\theta &= 0 \\ \frac{d(s\Omega)}{ds} + N_\theta \tan\alpha &= 0 \\ \frac{d(sM_\phi)}{ds} - M_\theta &= s\Omega \end{aligned} \qquad (5)$$

We immediately have, P being a constant,

$$\begin{aligned} s\Omega + sN_\phi \tan\alpha &= P \\ N_\phi &= -\Omega\cot\alpha + \frac{P}{s}\cot\alpha \\ N_\theta &= -\frac{d(s\Omega)}{ds}\cot\alpha \end{aligned} \qquad (6)$$

On the other hand, a simple calculation leads to the following expressions for the N's and the M's:

$$
\begin{aligned}
M_\phi &= K\left(w'' + \frac{\nu}{s} w' \right) \\
M_\theta &= K\left(\frac{w'}{s} + \nu\, w'' \right)
\end{aligned}
\qquad (7)
$$

$$
\begin{aligned}
N_\phi - \nu N_\theta &= D(1-\nu^2)v' \\
N_\theta - \nu N_\phi &= D(1-\nu^2)\frac{v + w\tan\alpha}{s}
\end{aligned}
\qquad (8)
$$

where ' designates differentiation with respect to s. Substituting the values (7) into equation (5_3), we have,

$$K(sw''' + w'') - \frac{Kw'}{s} = sQ \qquad (9)$$

If, on the other hand, we eliminate v between equations (8), we have,

$$s(N'_\theta - \nu N'_\phi)' + (1+\nu)(N_\theta - N_\phi) - D(1-\nu^2)w'\tan\alpha = 0$$

Let us replace the N's by means of formulae (6), and let $sQ = T$, we obtain,

$$sT'' + T' - \frac{T}{s} = -\frac{P}{s} - D(1-\nu^2)w'\tan^2\alpha \qquad (10)$$

Then, equation (9) takes the form,

$$sw''' + w'' - \frac{w'}{s} = \frac{T}{K} \qquad (11)$$

The elimination of w' between (10) and (11) is now easy. Let

$$L_1(\ ) = s\frac{d^2}{ds^2}(\ ) + \frac{d}{ds}(\ ) - \frac{1}{s}(\ )$$

we have,

$$w' = -\frac{P\cot^2\alpha}{D(1-\nu^2)}\frac{1}{s} - \frac{\cot^2\alpha}{D(1-\nu^2)} L_1(T) \qquad (12)$$

but, we see immediately that,

$$L_1\left(\frac{1}{s}\right) = 0$$

such that, by substituting w' into equation (11), we have

$$L_1L_1(T) = -\frac{D}{K}(1-\nu^2)\tan^2\alpha$$

Let

$$\delta^4 = \frac{D}{K}(1 - \nu^2) \tan^2 \alpha$$

the preceding equation can then be written as

$$L_1(L_1 T + i\delta^2 T) - i\delta^2 (L_1 T + i\delta^2 T) = 0$$

such that, by the same method already used, we are left to resolve the unique equation,

$$L_1(T) + i\delta^2 T = 0 \qquad (13)$$

which, due to the presence of imaginaries, will provide not only two but four particular solutions to the linear equation to be resolved.

By a change of variables

$$\eta = 2\delta\sqrt{i\tan\alpha}\,\sqrt{s} = y\sqrt{i}$$

equation (13) leads to the Bessel equation

$$\frac{d^2T}{d\eta^2} + \frac{1}{\eta}\frac{dT}{d\eta} + \left(1 - \frac{4}{\eta^2}\right) T = 0 \qquad (14)$$

It is precisely this equation which allows Flugge to write the detailed solution found on page 373 of his work. After the introduction of the usual Kelvin functions, elementary calculations lead to:

$$Q = \frac{1}{s}\begin{bmatrix} A_1(\mathrm{ber}\,y - \frac{2}{y}\mathrm{bei}'y) + A_2(\mathrm{bei}\,y + \frac{2}{y}\mathrm{ber}'y) \\ +B_1(\mathrm{ker}\,y - \frac{2}{y}\mathrm{kei}'y) + B_2(\mathrm{kei}\,y + \frac{2}{y}\mathrm{ker}'y) \end{bmatrix}$$

$N_\phi = -Q\cot\alpha$ (This condition becomes necessary if we assume essentially, that the cone is not truncated in the neighborhood of the peak.)

$$N_\theta = -\frac{\cot\alpha}{2s}\left\{\begin{aligned} & A_1(y\,\mathrm{ber}'y - 2\mathrm{ber}\,y + \frac{4}{y}\mathrm{bei}'y) + A_2(y\,\mathrm{bei}'y - 2\mathrm{bei}\,y - \frac{4}{y}\mathrm{ber}'y) \\ & +B_1(y\,\mathrm{ker}'y - 2\mathrm{ker}\,y + \frac{4}{y}\mathrm{kei}'y) + B_2(y\,\mathrm{kei}'y - 2\mathrm{kei}\,y - \frac{4}{y}\mathrm{ker}'y) \end{aligned}\right\}$$

$$M_\phi = \frac{2}{y^2}\left\{\begin{aligned} & A_1[\nu y\,\mathrm{bei}'y + 2(1-\nu)(\mathrm{bei}\,y + \frac{2}{y}\mathrm{ber}'y)] - A_2[\nu y\,\mathrm{ber}'y + 2(1-\nu)(\mathrm{ber}\,y - \frac{2}{y}\mathrm{bei}'y)] \\ & +B_1[\nu y\,\mathrm{kei}'y + 2(1-\nu)(\mathrm{kei}\,y + \frac{2}{y}\mathrm{ker}'y)] - B_2[\nu y\,\mathrm{ker}'y + 2(1-\nu)(\mathrm{ker}\,y - \frac{2}{y}\mathrm{kei}'y)] \end{aligned}\right\}$$

$$M_\theta = \frac{2}{y^2}\left\{\begin{array}{l} A_1[\nu y\,\mathrm{bei}'y + 2(1-\nu)(\mathrm{bei}\,y + \frac{2}{y}\mathrm{ber}'y)] - A_2[\nu y\,\mathrm{ber}'y + 2(1-\nu)(\mathrm{ber}\,y - \frac{2}{y}\mathrm{bei}'y)] \\ + B_1[\nu y\,\mathrm{kei}'y + 2(1-\nu)(\mathrm{kei}\,y + \frac{2}{y}\mathrm{ker}'y)] - B_2[\nu y\,\mathrm{ker}'y + 2(1-\nu)(\mathrm{ker}\,y - \frac{2}{y}\mathrm{kei}'y)] \end{array}\right\}$$

We leave aside, for the time being the difficulties existing around the peak of the cone for $s = 0$, and examine the validity of the approximations made at the beginning of the theory, namely, if, as indicated on page 3 the expression

$$S = K(v\cot\alpha + w)\frac{\tan^2\alpha}{s^2}$$

otherwise written as,

$$S = K\frac{\tan\alpha}{s}\frac{N_\theta - \nu N_\phi}{D(1-\nu^2)}$$

is truly negligible with respect to M_θ .

We know that for large values of s, the Kelvin functions have asymptotic expansions that can be easily handled. By using these expansions, we can give simple values to M and N (see Flugge, p. 374). The following evaluations result.

First, we easily perceive that the terms having the coefficients A dominate in the above expressions: This is of no great importance, however, since the considerations which follow maintain the same significance if we retain the terms having the coefficients B. We shall neglect these latter in order not to clutter our equations.

For large values of s , we use the appropriate approximations for the Kelvin functions and find,

$$S \simeq \frac{K\tan\alpha}{sD(1-\nu^2)}\left\{\begin{array}{l} -\frac{\cot\alpha}{2\sqrt{2\pi}}\frac{\sqrt{y}\,e^{y/\sqrt{2}}}{s}[A_1\cos(\frac{y}{2}+\frac{\pi}{8}) + A_2\sin(\frac{y}{2} + \frac{\pi}{8})] \\ + \frac{\nu\cot\alpha}{\sqrt{2\pi y}\;s}e^{y/\sqrt{2}}[A_1\cos(\frac{y}{2}-\frac{\pi}{8}) + A_2\sin(\frac{y}{2}-\frac{\pi}{8})] \end{array}\right\}$$

and the corresponding estimation of M_θ becomes,

$$M_\theta \approx \frac{2e^{y/\sqrt{2}}\,\nu}{y\sqrt{2\pi y}}\left[A_1 \sin\left(\frac{y}{2}+\frac{\pi}{8}\right) - A_2 \cos\left(\frac{y}{2}+\frac{\pi}{8}\right)\right]$$

We see that the first of these expressions is of the order of $\frac{K}{D} s^{-7/4} e^{y/\sqrt{2}}$, that is, substituting the values of the coefficients K and D, of the order,

$$h^2 s^{-7/4} e^{y/\sqrt{2}}$$

while the second expression is of the order,

$$s^{-3/4} e^{y/\sqrt{2}}$$

Thus, at least for appreciable values of s, we find that the first expression is quite negligible with respect to the second. This validates the use of the method followed in the preceding paragraphs.

Still keeping the example of the cone, let us perform this same operation by the method developed on page 4 . We arrive at the following equation,

$$g^{\prime V} = a g''' + b g'' + c g' + d g + k$$

in which the coefficients have the values stipulated on page 6 . For the case at hand, we find for these coefficients the following values:

$$a = -\frac{4\cos\alpha}{r} ; = -\frac{4\cot\alpha}{s}; \quad b = 0 ; \quad c = 0 ;$$

$$d = -\frac{12(1-\nu^2)}{h^2 s^2} ; \quad k = -\frac{18p(1-\nu^2)}{Eh^3}\frac{\cot\alpha}{s}$$

such that the equation in g can be written as,

$$g^{\prime V} + \frac{4\cot\alpha}{s} g''' + \frac{12(1-\nu^2)}{h^2 s^2} g + \frac{18p(1-\nu^2)}{Eh^3}\frac{\cot\alpha}{s} = 0$$

This linear equation is easy to study. The particular solution

$$g_0 = -\frac{3}{2Eh} s\cot\alpha$$

is immediately obvious and we have only the homogeneous equation left to solve. For this latter, we can easily construct two solutions as power series of the form,

$$p_1 s + p_2 s^2 + \ldots\ldots\ldots + p_n s^n + \ldots\ldots$$

Letting

$$A = \frac{12(1-\nu^2)}{h^2} \qquad B = 4\cot\alpha$$

we obtain the condition,

$$A p_n + n(n+1)(n+2)(B+n-1)p_{n+2} = 0$$

Thus we obtain two solutions as power series in s, of the form,

$$g_1 = s + \sum_1^{\infty} (-1)^n \frac{A^n}{B(B+2)\ldots\ldots(B+2n-2)} \frac{s^{2n+1}}{1\cdot 2\cdot 3^2\cdot 4\cdot 5^2\cdots 2n(2n+1)}$$

and

$$g_2 = s^2 + \sum_1^{\infty} (-1)^n \frac{A^n}{(B+1)(B+3)\ldots(B+2n-1)} \frac{s^{2n+2}}{2\cdot 3\cdot 4^2\cdot 5\cdot 6^2\cdots(2n)^2(2n+1)(2n+2)}$$

A general solution, then, is,

$$g = g_0 + \lambda g_1 + \mu g_2$$

Since the other solutions of the equation in g have a discontinuity for $s = 0$, the above solution becomes the only acceptable one for our example. It is now easy to formulate the quantities,

$$\frac{N_\theta - \nu N_\phi}{Es} \tan\alpha \quad \text{and} \quad \frac{g}{s}$$

which are to be compared, and to not if the first one is negligible with respect to the second one. The numerical calculations are long but without complications. The formulas for the N's are given on page 5. We can assign arbitrary values to the coefficients λ and μ, or, choose them such as to satisfy imposed boundary conditions. From these computations which we have outlined here but shall not reproduce in full detail, we find that the qpproximation is valid.

From what has preceded, we arrive at the following observation. The method based on the function g is more advantageous than the method, which we designate as classic, initially exposed. Although this latter method gives,i in the case of a cone, an explicit solution composed entirely of known functions, consequently resulting in four elementary solutions to the fourth order differential equation which is the center of the problem, the method in g gives simpler calculations introducing only two very simple series (much easier to calculate than the Kelvin functions) while rejecting the two elementary solutions which become infinte for s = 0 since the geometric significance of g forbids it to have large values.

From the point of view of adequacy of approximations, however, both methods are equally acceptable.

Let us now examine in greater detail what happens around the peak of the cone (for s = 0) where difficulties seem to be manifested. We shall see that these difficulties are only apparent and at the same time we shall see why we can neglect the terms affected by the coefficients B in the formulas on pages 12 and 13.

In the classical method we have the formulas,

$$\text{ber } y = 1 - \frac{(y/2)^4}{(2!)^2} + \frac{(y/2)^8}{(4!)^2} - \cdots\cdots\cdots$$

$$\text{bei } y = \frac{(y/2)^2}{(1!)^2} - \frac{(y/2)^6}{(3!)^2} + \frac{(y/2)^{10}}{(5!)^2} + \cdots\cdots$$

which are approximately

$$\text{ber } y \approx 1 \;, \quad \text{bei } y \approx \frac{y^2}{4}$$

for small values of y. Next, we have,

$$\text{ker } y = -(\text{ber } y)\log\frac{y}{2} + \frac{\pi}{4}\text{bei } y + \sum_{0}^{\infty} (-1)^m \frac{(\frac{y}{2})^{4m}}{(2m!)^2} \psi(2m+1)$$

$$\text{kei } y = -(\text{bei } y)\log\frac{y}{2} - \frac{\pi}{4}\text{ber } y + \sum_{0}^{\infty} (-1)^m \frac{(\frac{y}{2})^{4m+2}}{[(2m+1)!]^2} \psi(2m+2)$$

with

$$\psi(1) = -\gamma$$

$$\psi(m) = \frac{1}{1} + \frac{1}{2} + \ldots\ldots + \frac{1}{m} - \gamma$$

where γ is the Euler constant.

With the aid of theses formulas, it is easy to see that in the expressions for the functions Q, N, and M on pages 12 and 13, the parts affectedf by the coefficients B become infinite for $s = 0$, such that we must require the coefficients B to be zero. The parts affected by the coefficients A provide the following values as s goes to zero:

$$N_\phi = -\frac{A_2 \delta^2}{2} \quad ; \quad N = -A_2 \delta^2$$

$$M_\phi = M_\theta = A_1 \frac{1-\nu}{2}$$

We now see why we have neglected the terms in B in the preceding development since these vanish for the case of the conical shell.

We rework the same problem now by using the method of the function g. We shall see how this more direct method gives extraordinarily simpler calculations than the above method.

First, as on page 15, the value of g to be considered,

$$g = \left\{ \begin{array}{l} -zs\dfrac{\cot\alpha}{2Eh} + \lambda\left[s + \displaystyle\sum^{\infty} \frac{(-1)^n \; A^n \; s^{2n+1}}{B(B+2)-(B+2n-2)\cdot 1\cdot 2\cdot 3^2\cdot 4\cdot\cdot(2n+1)} \right] \\[2ex] + \mu\left[s^2 + \displaystyle\sum^{\infty} \frac{(-1)^n \; A^n}{(B+2)-(B+2n-1)} \; \frac{s^{2n+2}}{2\cdot 3\cdot 4^2\cdot\cdots(2n+2)} \right] \end{array} \right\}$$

is immediately free of terms which become infinite for $s = 0$. Next, we must substitute the function g into formulas giving the N's and the M's, given on page 5. We immediately see that in spite of the presence of denominators, the expressions g/s, g'/s, $-g/s^2$, remain finite, so that the value $s = 0$ introduces no difficulty. At this point, we obtain the following values:

$$N_\phi = \frac{Eh^2 \mu \cot a}{12(1-\nu^2)} \quad ; \quad N_\theta = \frac{5\, Eh^2 \mu \cot a}{12(1-\nu^2)}$$

$$M_\phi = M_\theta = \frac{Eh^3}{12(1-\nu)}\left(\frac{3\cot a}{2Eh} - \lambda\right)$$

Let us suppose, for example, that the cone is held in a cylindrical pipe such as to constrain the motion of its peak only along its axis. Then, since the unit of expansion of a parallel equals,

$$\frac{1}{E}(N_\theta - \nu N_\phi)$$

we see that the quantity $N_\theta - \nu N_\phi$ must be zero for $s = 0$, and for $s = s_1$ (= the length of a ray of the cone.) The first of these conditions requires that the coefficient μ be zero, and the second condition, for $s = s_1$, will be a linear expansion with respect to λ, so that the corresponding function g will be completely determined. All other cases of particular boundary conditions whould be treated in a similar fashion.

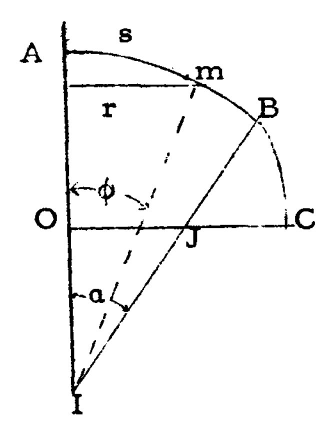
APPLICATIONS OF THE METHOD OF THE FUNCTION "g".

In this section, we should like to demonstrate the advantages of the method we have just described and which we designate as the method of the function g, for concrete applications. We shall measure this advantage by working a particular problem. In any respect, only a few special cases are of interest since one is limited by physically realizable constructions.

Thus, we are led to use, for shells of revolution, meridian* that take the form of the curve of anaarch, that is, two joined circumferences, or , the profile of a legthwise slice of a truncated cone.

Let us concentrate our attention on the case of the meridian as an arch. We shall begin by examining the solution as given by the method of the function g. We have already shown how to study the case of a given arc; the difficulty here will be in passing from the arc of one curve to that of another not havine the same analytical definition.

Actually, the meridian is composed of two arcs of circles, whose centers are I and J, as shown in the adjoining sketch. We can determine the dimensions of the figure by giving the legths OA and OC; we can also define, if need be, the angle α such that the point J will be in the middle of IB.



This stated, the problem of the deformation becomes, as we have already seen, the integration of a linear differential equation of the form,

$$g^{IV} = ag''' + bg'' + cg' + dg + k$$

where the coefficients a, b, c, d, and k have known values on the arc AB, and different values, also known, on arc BC.

Let us first study arc AB. The solution corresponding to the differential dequation will depend linearly on four constants, of which two will be determined by initial conditions. Let us call the remaining two constants λ and μ; these will be determined subsequently.

We recall the values of the coefficients a, b, c, d, and k on the arc AB to be,

$$a = -\frac{2\cot\phi}{R_1} ; \quad b = \frac{3 - \sin^2\phi}{R_1^2 \sin^2\phi}$$

$$c = -\frac{\cot\phi}{R_1^3 \sin^2\phi} (3 + 2\sin^2\phi)$$

$$d = -\frac{12(1 - \nu^2)}{h^2 R_1^2} + \frac{\cos^2\phi (3 + \sin^2\phi), + \nu^2 \sin^4\phi}{R_1^4 \sin^4\phi}$$

$$k = -\frac{36 (1 - \nu^2)}{Eh^3} \frac{\cos\phi}{r}$$

where R_1 designates the radius IA and h is the thickness of the shell.

We shall integrate the corresponding differential equation by letting the arc s vary from 0 (point A) to s_1 ($= R_1 \alpha$) at point B.

We shall not dwell here on the complication resulting from the fact that the coefficients a, b, c, and d become discontinuous for s = 0 , since the products as, bs^2, cs^3, ds^4 remain finite; we can apply to the equation the method used for the classic Euler equation.

Next, we must pass on to arc BC, that is to say, we must operate on a new differential equation. In the coefficients a, b, c, and d,RR_1 must be replaced by R_2, adn the angle ϕ must vary from α to π 2 . The arc s will have the value $R_1\alpha + R_2(\phi - \alpha)$ from B. If so desired, we can arrange to let $R_1 = 2 R_2$.

The difficulty is to know what initial values to adopt for g,

g', g'', and g''' corresponding to point B in the new equation. Let us note, in effect, that if we had on AB,

$$g' = \frac{dg}{ds} = \frac{1}{R_1}\frac{dg}{d\phi} \qquad \text{with } r = R_1 \sin\phi$$

on BC we now have,

$$g' = \frac{1}{R_2}\frac{dg}{d\phi} \qquad \text{with } r = OJ + R_2 \sin\phi$$

The coefficients of the differential equation undergo a discontinuity. What will be the consequences for the quantities g, g', g'', and g''' ? Since g represents the variation of the polar angle, it must necessarily remain continuous for the physical reason that we can allow no break in the curve. We must now refer to the formulas defining the strains and the moments. Those representing the moments are of the form,

$$M_1 = H\left(\frac{\cos\phi}{r} g + \nu g'\right)$$

$$M_2 = H\left(g' + \nu \frac{\cos\phi}{r} g\right)$$

where H is a constant. These moments must be continuous at B. Since g remains the same, g' (that is dg ds) must retain the same value; this is further required by the fact that the meridian curve must not have angular points.

Next, the formulas for the constraints in the principal directions (as established in our quarterly status report no.) are as follows:

$$N_1 = \frac{pr_1}{2h\sin\phi} + H_1\left[\cot\phi\, g_1'' + \frac{\cos^2\phi}{\sin\phi}\frac{g_1'}{r_1} - \left(\frac{\cos^3\phi}{r_1^2\sin\phi} + \nu\frac{\cos\phi}{R_1 r_1}\right)g_1\right]$$

$$N_2 = \frac{pr_1}{h\sin\phi}\left(1 - \frac{r_1}{2R_1\sin\phi}\right) + H_1\left[\frac{r_1}{\sin\phi} g_1''' + 2\cot\phi\, g_1'' - \frac{\cos^2\phi}{\sin\phi}\frac{g_1'}{r_1} + \frac{\cos^3\phi}{\sin\phi\, r_1^2} g_1\right.$$

$$N_3 = H_1 [g_1'' + g_1' \frac{\cos\phi}{r_1} - \frac{\cos^2\phi}{r_1^2} g_1]$$

for the arc AB; for the arc BC, we replace the index 1 by the index 2 in the above equations.

The main point is that these expressions are linear functions of g, g' and g'' for N_1 and N_3, and also of g''' for N_2. By stipulating that for $\phi = \alpha$, the values of N_2 and N_3 are to remain unchanged, we obtain two linear equations which will determine g_2'' and g_2'''. We may fear arriving at an impossibility by stipulating that N_1 not have any discontinuity. However, it is easy to show that this will not occur, and that the values already obtained for the constants assure the continuity of N_1. Moreover, this is further true from the fact that among the formulae which led to the determination of the values of the N, s is the following,

$$\frac{pr}{2h} + N_3 \cos\phi - N_1 \sin\phi = 0$$

It is thus obvious that N_1 will be continuous for $\phi = \alpha$ if N_3 is.

We now have all that is necessary to pass from the arc AB of the meridian to the arc BC on which the nemerical integration of the equation in g will be performed as it was on the first arc but with new values for the coefficients.

In all that has preceded, we have left the initial values $g_1'(0)$, and $g_1'''(0)$ indeterminate. It is useful to know how to define these constants. First, it is clear that the solution in g which holds on the first arc is of the form,

$$g(s) = P(s)\lambda + Q(s)\mu + R(s)$$

To define the functions R, Q, and R, it is sufficient to undergo the calculations described above by adopting the following initial conditions:

for P: let k = 0 in the equation in g and calculate the solution corresponding to $\lambda = 1$, $\mu = 0$.

for Ω: let $k = 0$ and assume initially the conditions $\lambda = 0$, $\mu = 1$.

for R: take the full equation with k having the value given on page 20 and take as initial conditions $\lambda = 0$, $\mu = 0$.

The constants λ and μ figure linearly in all the equations considered up to this point. Their precise determination will ultimately depend on the boundary conditions imposed on the system. If, for example, the shell is constrained at C by a sleeving which forbids all expansion towards the exterior, g and g' will have to be zero at C. This will give two linear equations for λ and μ which will determine them. Further belaboring of this point seems unnecessary.

Having obtained these developments, let us next study the same problem, the meridian as an arch, by using the so called classical method

We shall note a singular particularity, namely, that although at the beginning of these calculations we find classical functions, in this case, hypergeometric functions, which avoid the problem of numerical . integration, at least at first, this advantage becomes ultimately outweighed by an embarrassing difficulty.

Let us start with the equations of equilibrium of the shell. With the usual notations found in the work of W. Flugge, these equations for the arc AB of the meridian are as follows:

$$\frac{d}{d\phi}(N_{\phi} \sin\phi) - N_{\theta} \cos\phi - Q_{\phi} \sin\phi = - p_{\phi} \sin\phi$$

$$\frac{d}{d\phi}(Q_{\phi} \sin\phi) + N_{\theta} \sin\phi + N_{\phi} \sin\phi = p_{r} \sin\phi \tag{15}$$

$$\frac{d}{d\phi}(M_{\phi} \sin\phi) - M_{\theta} \cos\phi = a Q_{\phi} \sin\phi$$

Eliminating N_{θ} between the first two equations gives,

$$\frac{d}{d\phi} (N_{\phi} \sin^2\phi + Q_{\phi} \sin\phi \cos\phi) = \sin\phi \, (- p_{\phi} \sin\phi + p_{r} \cos\phi)$$

from which, letting P be a constant of integration,

$$N_{\phi} = -Q_{\phi} \cot\phi + \frac{P}{\sin^2\phi} - \frac{1}{\sin^2\phi} \int \sin\phi \, (p_{\phi} \sin\phi - p_{r} \cos\phi) d\phi \tag{16}$$

and consequently,

$$N_{\theta} = - \frac{\partial Q_{\phi}}{\partial\phi} + p_{r} + \frac{1}{\sin^2\phi} \int \sin\phi \, (p_{\phi} \sin\phi - p_{r} \cos\phi) d\phi - \frac{P}{\sin^2\phi} \tag{17}$$

We know that the constraints N are related to the small deformations u, v, and w by the formulas,

$$v' + w = -\frac{a}{D(1-\nu^2)}(N_\phi - \nu N_\theta)$$

$$v\cot\phi + w = \frac{a}{D(1-\nu^2)}(N_\theta - \nu N_\phi)$$

Let us differentiate the second of these equations with respect to ϕ and eliminate v' and w at the expense of v and w' among the three relations thus obtained. After several elementary calculations, we have,

$$w' - v + \frac{a}{D(1-\nu)}(N_\phi - N_\theta)\cot\phi = \frac{a}{D(1-\nu^2)}(N'_\theta - \nu N'_\phi)$$

But if we let,

$$w' - v = a\chi$$

this becomes,

$$\chi = \frac{a}{D(1-\nu^2)}[N'_\theta - \nu N'_\phi - (1+\nu)(N_\phi - N_\theta)\cot\phi] \qquad (18)$$

Let us now substitute the N's from formulas (16) and (17) into this equation. After a few easy transformations, we have,

$$D(1-\nu^2)\chi = -\Omega''_\phi - \Omega'_\phi\cot\phi + \Omega_\phi(\cot^2\phi - \nu) + (1+\nu)p_\phi + p'_r \qquad (19)$$

On the other hand, the moments M can be written as,

$$M_\phi = \frac{K}{a}(\chi' + \nu\chi\cot\phi)$$

$$M_\theta = \frac{K}{a}(\chi\cot\phi + \nu\chi')$$

such that by eliminating the M's among these two equations and equation (15-3), we have,

$$\chi'' + \chi'\cot\phi - (\cot^2\phi + \nu)\chi = \frac{a^2}{K}\Omega_\phi$$

As usual, let,

$$L(\ ) = (\ )'' + (\ )'\cot\phi - (\ )\cot^2\phi$$

Equations (18) and (19) become,

$$L(\chi) - \nu\chi = \frac{a^2}{K} Q_\phi$$

$$L(Q_\phi) + \nu Q_\phi = - D(1 - \nu^2) \quad + (1 + \nu)p_\phi + p'_r$$

Eliminating χ between these last two equations leads to,

$$L\,L\,Q_\phi - \nu^2 Q_\phi = - \frac{Da^2}{K}(1 - \nu^2)Q_\phi + (1 + \nu)(Lp_\phi - \nu p_\phi) + Lp'_r - \nu p'_r \quad (2$$

Let us apply these results to the case of constant external pressure equal to p. In the above formulas, we must make $p_\phi = 0$ and $p_r = p$. Equation (20) defining Q_ϕ is then the same as that of W. Flugge, and the formulas defining the N's become,

$$N_\phi = - Q_\phi \cot\phi + \frac{P}{\sin^2\phi} - \frac{p}{2}$$

$$N_\theta = - Q'_\phi + \frac{p}{2} - \frac{P}{\sin^2\phi}$$

with

$$L\,L\,Q_\phi - \nu^2 Q_\phi = - \frac{Da(1 - \nu^2)}{K} Q_\phi$$

for Q_ϕ. However, we know that if we let

$$x = \cos^2\phi \quad ; \quad Q_\phi = z\sin\phi$$

the equation in Q_ϕ becomes a hypergeometric equation

$$\frac{d^2z}{dx^2} + \frac{\gamma - (1 + \alpha + \beta)x}{x(1 - x)}\frac{dz}{dx} - \frac{\alpha\beta}{x(1-x)}z = 0$$

where the constants α, β , and γ have the following values,

$$\alpha = \frac{1}{4}[3 + \sqrt{5 + 8i\kappa^2}]$$

$$\beta = \frac{1}{4}[3 - \sqrt{5 + 8i\kappa^2}] \; ; \qquad \gamma = \frac{1}{2}$$

where we have let,

$$\kappa^4 = \frac{D(1 - \nu^2) a^2}{4K} - \frac{\nu^2}{4}$$

Let us move along the arc AB of the meridian. At the peak A, we have $\phi = 0$ and $x = 1$. This point is one of three singular p points in the hypergeometric equation. Furthermore, for this latter, the quantity, $\alpha + \beta - \gamma$, is a whole number. We are thus in a case where the use of hypergeometric functions poses special analytical difficulties. We must define two particular solutions of the equation which are regular in the neighborhood of the point $x = 1$, and we must use these solutions on the arc AB.

We shall not develop the point of defining a particular solution on arc AB because we shall see in what follows that is is completely unnecessary to obtain such precision here.

In effect, we shall have to join the solution suitable on arc AB with that suitable on arc BC. In order to obtain this latter one, we will have to integrate formulas analogous to equations (15), (16),.. and following, but written for the arc BC.

Let us write these new equations. If b is the new radius of curvature, the equilibrium conditions for the shell are:

$$\frac{d}{d\phi}(rN_\phi) - bN_\theta \cos\phi - rQ_\phi = - brp_\phi$$

$$\frac{d}{d\phi} (rQ_\phi) + bN_\theta \sin\phi + rN_\phi = brp_r$$

$$\frac{d}{d\phi}(rM_\phi) - bM_\theta \cos\phi = brQ_\phi$$

and the formulas defining the N's and the M's now are,

$$N_\phi = D\left(\frac{v' + w}{b} + \nu \frac{v\cos\phi + w\sin\phi}{\delta + b\sin\phi}\right)$$

$$N_\theta = D\left(\frac{v\cos\phi + w\sin\phi}{\delta + b\sin\phi} + \nu \frac{v' + w}{b}\right)$$

$$M_\phi = \frac{K}{b}\left(\frac{w''}{b} + \nu \frac{w'\cos\phi}{\delta + b\sin\phi}\right)$$

$$M_\theta = \frac{K}{b}\left(\frac{w'\cos\phi}{\delta + b\sin\phi} + \nu \frac{w''}{b}\right)$$

since the radius vector is

$$r = \delta + b\sin\phi$$

and the principal radii of curvature of the shell are,

$$R_1 = b \; ; \quad R_2 = b + \frac{\delta}{\sin\phi}$$

From these equations, we must try to derive consequences analogous to those which we have deduced from the equations concerning arc AB. However, the results obtained above strongly suggest that the problem here is parctically inextricable. By this method it seems impossible to obtain an explicit solution.

This demonstrates that the method based on the intervention of the function g has a considerable advantage over the classical method since no such difficulty occurs with the former method when passing from arc AB to arc BC.

For this reason, in what follows, we shall use the method of the function g.

CORRESPONDANCE BETWEEN CURVED SHELLS AND PLANAR SHELLS.

EXAMPLE 1: OGIVES OF REVOLUTION

The theory we have developed can be used in numerous applications. We shall elaborate on two which seem to be of particular practical interest.

In the construction of ogives of revolution (such as would be used as satellites for interplanetary experiments), the strength of the structure is obviously of the greatest importance. This strength is related to the form of the meridian of the shell of revolution. The shell is generally assumed to be of constant thickness.

In order to simplify the preliminary experiments in the building of a model of the ogive of revolution, it would be interesting to try the following method. We should like to establish a correspondance between the shell to be studied and another shell of simpler form, for example planar, but having a variable thickness. The correspondance should be such that two of the constraint components be equal respectively at the corresponding points M and m of the two shells. Since there are more than two functions to be determined, this will not assure identity of the state of the constraints in the two shells. However, if two of the components, for example, N_ϕ and N_θ are made equal, it will be easy to compare those components (in this case the moments), which will not have been made equal a priori. We would then have a precise idea of the approximations obtained in replacing the curved shell by the planar shell. The variable thickness of this latter could be experimentally modified so as to give the least dangerous values possible to the constraints. The results would next be applied to the curved shell whose profile is obtained by the correspondance established between the two shells.

In this calculation we have two unknown functions. One is the expression,

$$\sigma = f(s)$$

which establishes the relation, unknown beforehand, between the curvilinear abscissas, σ and s, of the two shells; the other is the unknown function,

$$h = G(s)$$

which defines the thickness of the planar shell at the point s (s is then, the distance to the center).

By means of the method of the function g , we shall determine, if need be, by numerical analysis, the values of $N_\phi(\sigma)$, $N_\theta(\sigma)$, $M_\phi(\sigma)$, and $M_\theta(\sigma)$ which correspond to the value σ of the arc of the meridian, or, to put it in other words, to the angle ϕ between the normal to the shell and its axis of revolution.

The problem having been stated, we shall now study the case of a planar disc of variable thickness. We assume, of course, that the constants of elasticity are the same as those for the curved shell, that is to say, both objects are made from the same metal. The usual equilibrium equations and the formulas defining the N's and the M's, are, for the general case:

$$\frac{d}{d\phi}(rN_\phi) - R_1 N_\theta \cos\phi - rQ_\phi = -rR_1 p_\phi$$

$$\frac{d}{d\phi}(rQ_\phi) + R_1 N_\theta \sin\phi + rN_\phi = rR_1 p_r$$

$$\frac{d}{d\phi}(rM_\phi) - R_1 M_\theta \cos\phi = rR_1 Q_\phi$$

$$N_\phi = D\left[\frac{\frac{dv}{d\phi} + w}{R_1} + \nu\,\frac{v\cos\phi + w\sin\phi}{r}\right]$$

$$N_\theta = D\left[\frac{v\cos\phi + w\sin\phi}{r} + \nu\,\frac{\frac{dv}{d\phi} + w}{R_1}\right]$$

$$M_\phi = \frac{K}{R_1}\left[\frac{d}{d\phi}\left(\frac{1}{R_1}\frac{dw}{d\phi} \right) + \nu \frac{\frac{dw}{d\phi}\cos\phi}{r} \right]$$

$$M_\theta = \frac{K}{R_1}\left[\frac{\frac{dw}{d\phi}\cos\phi}{r} + \nu \frac{d}{d\phi}\left(\frac{1}{R_1}\frac{dw}{d\phi} \right) \right]$$

If R_1, the radius of curvature of the meridian, becomes infinite -- recalling that $R_1 d\phi = ds$ -- the above equations become:

$$\frac{d}{ds}(rN_\phi) - N_\theta = -rp_\phi$$

$$\frac{d}{ds}(rQ_\phi) = rp_r \qquad (21)$$

$$\frac{d}{ds}(rM_\phi) - M_\theta = rQ_\phi$$

$$N_\phi = D\left(\frac{dv}{ds} + \nu \frac{v}{r} \right)$$

$$N_\theta = D\left(\frac{v}{r} + \nu \frac{dv}{ds} \right)$$

$$M_\phi = K\left(\frac{d^2w}{ds^2} + \frac{\nu}{r}\frac{dw}{ds} \right) \qquad (22)$$

$$M_\theta = K\left(\frac{1}{r}\frac{dw}{ds} + \nu \frac{d^2w}{ds^2} \right)$$

Thus, for a planar disc of constant thickness, if we substitute the values from above into the equilibrium equations, we obtain,

$$(sw'' + \nu w')' - \left(\frac{w'}{r} + \nu w'' \right) = rQ_\phi$$

$$rQ_\phi = \text{constant} = A$$

$$(sv' + \nu v)' - (\frac{v}{s} + \nu\nu v') = 0$$

that is,

$$r\Omega_\phi === A$$

$$sv'' + v' - \frac{v}{s} = 0$$

$$sw''' + w'' - \frac{w'}{r} = A$$

From the above, we can determine the unknowns, Ω_ϕ, v, and w.

For the case of thickness varying with s, since we have

$$D = \frac{Eh}{1 - \nu^2} \quad ; \quad K = \frac{Eh^3}{12(1 - \nu^2)}$$

the same calculation gives rise to the following equations:

$$\frac{d}{ds}[h (sv' + \nu v)] - h (\frac{v}{s} + \nu v') = 0 \tag{23}$$

that is,

$$h' (sv' + \nu v) + h (sv'' + v' - \frac{v}{s}) = 0$$

and also,

$$\frac{d}{ds} [h^3 (sw'' + \nu w')] - h^3 (\frac{w'}{s} + \nu w'') = A$$

that is,

$$3h^2h' (sw'' + \nu w') + h^3 (sw''' + w'' - \frac{w'}{s}) = A \tag{24}$$

Suppose, then, that at corresponding points, the values N_ϕ and N_θ are equal for both shells. For the curved shell, N_ϕ and N_θ have the known values $F(\phi)$ and $G(\phi)$ as a function of the angle of inclination, ϕ, and we write,

$$\frac{1 - \nu^2}{E} N_\phi = h (v' + \nu \frac{v}{s}) = F(\phi) \frac{1 - \nu^2}{E} \tag{25}$$

$$\frac{1-\nu^2}{E} N_\theta = h\left(\frac{v}{s} + \nu v'\right) = G(\phi)\frac{1-\nu^2}{E} \qquad (26)$$

so that equation (23)becomes,

$$\frac{d}{ds}\left[sF(\phi)\right] - G(\phi) = 0 \qquad (27)$$

We consider that this equation establishes the relation for which we were searching, between the abscissa s on the planar disc and angle ϕ of the ogive curve (ϕ and σ are equivalent here). This relation can be written,

$$\frac{ds}{s} = \frac{F'(\phi)\, d\phi}{G(\phi) - F(\phi)}$$

thereby giving s as a function of ϕ through a quadrature.

This relation having been obtained, equations (25) and (26) give,

$$\frac{v' + \nu \frac{v}{s}}{\frac{v}{s} + \nu v'} = \frac{F(\phi)}{G(\phi)}$$

from which we find v by integrating the linear equation giving $\frac{v'}{v}$ as a function of s.

Next, either equation (25) or (26) will give the desired value of the thickness h as a function of s. Further, equation (24) will become a differential equation in w (equation which is otherwise linear) to yield this last unknown.

EXAMPLE 2 : TRUNCATED CONE SHELL.

Along the same lines of reasoning as in the example of the ogive of revolution, let us now consider an equally interesting case that of a truncated cone shell and let us determine what would be the characteristcs of a corresponding planar disc having a hole in the center.

The formulas to be used are obviously related to those we have developed on pages 11, 12 ,and 13. However, the analysis must be modified since we no longer have the presence of the peak of the cone, to take into account,

We recall the formulas for the strains, N, and the moments, M:

$$N_s = D\left[\frac{dv}{ds} + \frac{\nu}{s}(v + w\tan\alpha) \right]$$

$$N_\theta = D\left[\frac{v + w\tan\alpha}{s} + \nu \frac{dv}{ds} \right] \qquad (28)$$

$$M_s = K\left[\frac{d^2w}{ds^2} + \frac{\nu}{s}\frac{dw}{ds} \right]$$

$$M_\theta = K\left[\frac{1}{s}\frac{dw}{ds} + \nu \frac{d^2w}{ds^2} \right] \qquad (29)$$

with

$$D = \frac{Eh}{1-\nu^2} \quad ; \quad K = \frac{Eh^3}{1-\nu^2}$$

On the other hand, the equilibrium equations are:

$$\frac{d}{ds}(sN_s) - N_\theta = 0$$

$$\frac{d}{ds}(sQ_s) + N_\theta \tan\alpha = ps \qquad (30)$$

$$\frac{d}{ds}(sM_s) - M_\theta = sQ_s$$

From these last equations, we have,

$$\frac{d}{ds}(sQ_s + sN_s \tan\alpha) = ps$$

that is,

$$sQ_s + sN_s \tan\alpha = \frac{ps^2}{2} + P$$

where P is an arbitrary constant. Thus, we have,

$$N_s = -Q_s \cot\alpha + \frac{P}{s}\cot\alpha + \frac{ps}{2}\cot\alpha \qquad (31)$$

On the other hand, equations (28) can be written,

$$\left.\begin{aligned} N_s - \nu N_\theta &= D(1-\nu^2)v' \\ N_\theta - \nu N_s &= D(1-\nu^2)\frac{v + w\tan\alpha}{s} \end{aligned}\right\} \qquad (32)$$

and equations (29) substituted into the last of (30), give

$$sQ_s = K\left(sw''' + w'' - \frac{w'}{s}\right) \qquad (33)$$

Let us now eliminate v between the two formulas of (32). We have,

$$s(N_s' - \nu N_\theta') + (1+\nu)(N_\theta - N_s) = D(1-\nu^2)w'\tan\alpha \qquad (34)$$

where ' designates differentiation with respect to the arc s. In this last formula, let us replace N_s by its value given in (31), and N_θ by the expression derived from (30)

$$N_\theta = \frac{d}{ds}\left(-sQ_s + P + \frac{ps^2}{2}\right)\cot\alpha \qquad (35)$$

Letting

$$T = sQ_s \qquad (36)$$

equation leads to the following result after some simplifications,

$$-sT'' - T' + \frac{T}{s} - \frac{P}{s} + \frac{p(1-2\nu)}{2}s = D(1-\nu^2)w'\tan^2\alpha \qquad (37)$$

Using the symbolic operator,

$$L = s(\ )'' + (\ )' - \frac{(\ )}{s}$$

we can write equations (33) and (37) in the form,

$$T = K\,L(w') \tag{38}$$

$$D(1-\nu^2)w'\tan^2\alpha = -L(T) - \frac{P}{s} + \frac{1-2\nu}{2}ps \tag{39}$$

Let us eliminate w' between these last two equations. Letting,

$$\delta^4 = \frac{D(1-\nu^2)\tan^2\alpha}{K}$$

we obtain,

$$\delta^4 T = -LLT - P\,L\left(\frac{1}{s}\right) + \frac{1-2\nu}{2}pL(s)$$

However, it is easy to see that we have,

$$L(s) = 0 \quad ; \quad L\left(\frac{1}{s}\right) = 0$$

Our final equation defining T is as follows,

$$L\,L(T) + \delta^4 T = 0 \tag{40}$$

We recognize this to be the same equation as that of W. Flugge on page 324 of his classic work. This equation can be written in the following forms:

$$L(LT + i\delta^2 T) - i\delta^2(LT + i\delta^2 T) = 0$$
$$L(LT - i\delta^2 T) + i\delta^2(LT - i\delta^2 T) = 0 \tag{41}$$

so that equation (40) possesses solutions of simpler equations: $LT \pm i\delta^2 T = 0$. Since the equation in T is linear, it is sufficient, thanks to the presence of the imaginary parts, to integrate one of the equations (41) in order to obtain four solutions of (40), and consequently to have the general solution.

As is well known, the change of variables

$$\eta = 2\delta\sqrt{is}$$

transforms the first equation (40) into the Bessel equation

$$\frac{d^2T}{d\eta^2} + \frac{1}{\eta}\frac{dT}{d\eta} + \left(1 - \frac{4}{\eta^2}\right)T = 0 \qquad (42)$$

such that, adopting the notationssfound in the work of Watson, equation (42) has as its integral,

$$T = A\,J_2(\eta) + B\,H^{(1)}_{(2)}(\eta)$$

Let

$$y = \frac{\eta}{\sqrt{i}}$$

and let us use the Themson formulas to separate the real and the imaginary parts. Thus,

$$J_2(\eta) = \frac{2}{y}\text{bei}'y - \text{ber}y + i\left(\frac{2}{y}\text{ber}'y + \text{bei}y\right)$$

$$H^{(1)}_{(2)}(\eta) = \frac{2}{\pi}\left(\frac{2}{y}\text{ker}'y + \text{kei}y\right) - \frac{2i}{\pi}\left(\frac{2}{y}\text{kei}'y - \text{ker}y\right)$$

which immediately gives the four solutions of (40)

Because of the definition (36) of T, we have for Ω_s the following expression,

$$\Omega_s = \frac{1}{s}\left\{\begin{array}{l} A_1\left(\text{ber}y - \frac{2}{y}\text{bei}'y\right) + A_2\left(\text{bei}y + \frac{2}{y}\text{ber}'y\right) \\ \\ + B_1\left(\text{ker}y - \frac{2}{y}\text{kei}'y\right) + B_2\left(\text{kei}y + \frac{2}{y}\text{ker}'y\right) \end{array}\right\}$$

where the A's and the B's are arbitrary constants.

We next have N_s by means of equation (31), then N_θ from equation (34). We solve for w by integrating

$$sw''' + w'' - \frac{w'}{s} = \frac{T}{K}$$

which is elementary since we have noted above that L(s) and $L(\frac{1}{s})$ are null.

We thus find,

$$w' = as + \frac{b}{s} + \frac{s}{2K}\int T ds - \frac{1}{2Ks}\int s^2 T ds$$

from which we find w and consequently v. Next, formulas (29) will give the M's.

The problem is then resolved. It entails seven arbitrary constants (not counting the constant of integration for w), namely, P A_1, A_2, B_1, B_2, a, and b. These constants will be determined from the boundary conditions imposed on the system. The only difficulties that could arise will be strictly of numerical computation nature.